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IRREGULAR NETWORKS, REGULAR GRAPHS AND INTEGER MATRICES WITH DISTINCT ROW AND COLUMN SUMS

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A network is a simple graph to which each edge has been assigned a positive integer weight. A network is irregular if the sum of the edges incident to each vertex is distinct. In this paper we study this concept for regular or nearly regular graphs and derive a relationship to integer matrices with distinct row and column sums.

In particular, we consider the parameter, $s(G)$, the irregularity strength of a graph G, which is the smallest maximum weight over all irregular networks with underlying graph G. It is known that if G is an r-regular graph of order n, then $s(G) \geq (n+r-1)/r$. We exhibit infinitely many r-regular graphs with $s(G) = \lceil (n+r-1)/r \rceil$, and it is proved that $s(G) \leq \lceil n/2 \rceil + 2$, for all r-regular graphs on n vertices if r is even.

We also study totally irregular matrices, that is positive integer matrices with distinct row and column sums having the smallest possible maximal entry. As a corollary, we can determine the strength of complete bipartite graphs $K_{p,q}$ except in the case when $p = q$ is odd.

1. Introduction

The study of regular graphs has been an extensive one dating back to Petersen [5]. In [1] the idea of studying structures where all the degrees are different was introduced. Except for the trivial one-point graph, there are no simple graphs which are irregular in this sense. For the purpose of this paper, a *network* is a simple graph to which each edge is assigned a positive integer value or *weight.* The *degree* of a vertex in a network is the sum of the weights of its incident edges. A network is *irregular* if all the vertices have distinct degrees. The *strength* of a network is the maximum weight assigned to any edge, while the *irregularity strength* of a graph G, denoted $s(G)$, is the minimum strength among irregular networks with underlying graph G.

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The main results in this paper are partitioned into three sections. In Section 2, a technique is presented to construct infinitely many irregular networks of strength at most $s+1$ from networks of strength s (Theorem 2.3). In particular, this enables us to find an infinite class of r-regular graphs of order n, with irregularity strength $\lceil n/r \rceil + 1$, which is essentially the lower bound given in [1] (Theorem 2.8). Additionally, as a simple corollary we obtain that 3 is the irregularity strength of complete multipartite graphs, with at least three parts and each part having the same number of vertices (Theorem 2.4).

In Section 3 we show that if G is 2-regular then $\lceil n/2 \rceil \leqslant s(G) \leqslant \lceil n/2 \rceil + 2$ (Theorem 3.4). This allows us to improve the bound of $n-1$ given in [4] and show that if G is an r-regular graph with r even or $r \geqslant n/2$, then $s(G) \leqslant [n/2] + 2$ (Theorems 4.5 and 4.6). Finally, in Section 4, we determine the irregularity strength for complete bipartite graphs $K_{p,q}$ with the exception of $K_{p,p}$ for p odd (Theorems 4.3 and 4.5).

Each section suggests a number of questions as well as directions for further study. We mention one of these questions as a vehicle to stimulate interest. This problem is what prompted the present study. To the authors' knowledge, no complete solution exists for determining $s(K_{2k+1,2k+1})$. An intriguing way to consider this problem is to decide an answer to the following question: Does there exist a $(2k+1) \times (2k+1)$ matrix with entries -1, 0 or 1, all of whose row sums and column sums are distinct?

2. Irregular networks on regular graphs obtained by vertex multiplication

Let G be an r-regular graph with t vertices and let $G^{(k)}$ denote the graph obtained from G by replacing each vertex of G by an independent set of k vertices, and including all edges between k sets if there was an edge in G. Note that $G^{(k)}$ is a kr-regular graph of order kt.

In Lemmas 2.1 and 2.2, procedures are described to obtain an irregular network on $G^{(k)}$ so that its strength is independent of k. Suppose that an irregular network of strength s is given on G. Let the vertex set of G be $\{1, \ldots, t\}$ and denote by $\omega(ij)$ the weight of edge ij. The adjacency matrix of the network is a $t \times t$ matrix W defined as follows:

$$W(i,j) = \begin{cases} 0 & \text{if } ij \notin E(G) \\ \omega(ij) & \text{otherwise.} \end{cases}$$

Obviously, W is a symmetric integer matrix with distinct row sums $d_1, d_2, \ldots, d_t$, with $r \leqslant d_i \leqslant rs$ for every i, $i = 1, \ldots, t$.

Lemma 2.1. *Assume W is the adjacency matrix of an irregular network of strength s with t vertices. Replace each zero entry of W by a $k \times k$ zero matrix and each* *entry* $W(p, q) \neq 0$ *by a* $k \times k$ *matrix* $W_{pq}^{(k)}$, *where*

$$W_{pq}^{(k)}(i, j) = \begin{cases} 1 & \text{if } i + j \leqslant k \\ W(p, q) & \text{if } i + j = k + 1 \\ s + 1 & \text{otherwise.} \end{cases}$$

Then the row sums of the resulting $tk \times tk$ *matrix* $W^{(k)}$ *are distinct, that is,* $W^{(k)}$ *is the adjacency matrix of an irregular network of strength* $s + 1$ *on* $G^{(k)}$.

Proof. Since every row of W contains r non-zero entries, each row sum d_i $(1 \leqslant i \leqslant t)$ is replaced in $W^{(k)}$ by the following sequence of k distinct row sums:

$$d_i + (k-1)r,\ d_i + (k-1)r + rs, \ldots, d_i + (k-1)r + (k-1)rs.$$

If, for some i, j, x and y

$$d_i + (k-1)r + xrs = d_j + (k-1)r + yrs,$$

then $d_i - d_j = (y - x)rs$.

Since $|d_i - d_j| < rs$, $y = x$ and $d_i = d_j$, which implies that $i = j$. Thus the row sums of $W^{(k)}$ are different. □

Lemma 2.2. *Assume* W *is the adjacency matrix of an irregular network of strength* s *with* t *vertices. Replace each zero entry of* W *by a* $k \times k$ *zero matrix and each entry* $W(p, q) \neq 0$ *by a* $k \times k$ *matrix* $W_{pq}^{(k)}$, *where*

$$W_{pq}^{(k)}(i, j) = \begin{cases} 1 & \text{if } i + j \leqslant k \\ W(p, q) & \text{if } i - j = k + 1 \\ s & \text{otherwise.} \end{cases}$$

If one of r *and* rs *is not a row sum of* W, *then the row sums of the resulting* $tk \times tk$ *matrix* $W^{(k)}$ *are distinct, that is,* $W^{(k)}$ *is the adjacency matrix of an irregular network of strength* s *on* $G^{(k)}$.

Proof. Each row sum d_i $(1 \leqslant i \leqslant t)$ is replaced by the following sequence of k distinct sums:

$$d_i + (k-1)r + xr(s-1): \quad x = 0, 1, \ldots, k-1.$$

Suppose that there are identical row sums in $W^{(k)}$, that is, for some i, x and j, y,

$$d_i - d_j = (y - x)r(s-1). \quad \text{Obviously, } y - x \leqslant 1.$$

In case $y - x = 1$, $\{d_i, d_j\} = \{rs, r\}$ which cannot happen by assumption. Hence $y = x$, so that $i = j$, a contradiction. □

Lemmas 2.1 and 2.2 can be incorporated into the following theorem on the irregularity strength of graphs obtained from regular graphs by vertex multiplication.

Theorem 2.3. *Let G be an r-regular graph, N an irregular network of strength s on G, and $G^{(k)}$ the graph obtained from G by vertex multiplication. Then $s(G^{(k)}) \leqslant s+1$. Moreover, if N does not contain a vertex of degree r or sr, then $s(G^{(k)}) \leqslant s$.*

As an application of Theorem 2.3, we determine the irregularity strength of complete t-partite graphs with k vertices in each vertex class, $t \geqslant 3$.

Theorem 2.4. *Let G be a complete (non-bipartite) multipartite graph with the same number of vertices in each vertex class. Then $s(G)=3$.*

Proof. If G is t-partite with k vertices in each part, then G is obviously isomorphic to $K_t^{(k)}$, where K_t denotes the complete graph of order t. In [1] it is shown that $s(K_t)=3$ for every $t \geqslant 3$. In fact, the upper left $t \times t$ matrix of the infinite array in Fig. 1 is the adjacency matrix of an irregular network of strength 3 on K_t so that either $t-1$ or $3(t-1)$ is not a row sum. Hence $s(G) \leqslant 3$ follows by Theorem 2.3. The inequality $s(G) \geqslant 3$ follows from the following general observation.

Proposition 2.5. *For every regular graph of order n, $n \geqslant 3$, $s(G) \geqslant 3$.*

Proof. Suppose on the contrary that G is an r-regular graph and N is an irregular network of strength 2 on G. Let $G = G_1 \cup G_2$ where G_i is the subgraph generated by all edges of weight i, $i=1$ and 2. Since $d_G(x) = d_{G_1}(x) + 2d_{G_2}(x) = r + d_{G_2}(x)$, each vertex of G_2 should have a different degree, a contradiction. □

Proposition 2.5 also follows for noncomplete graphs from the lower bound

$$s(G) \geqslant (n+r-1)/r \tag{1}$$

which is valid for every r-regular graph G of order n (see [1]). As an application of Theorem 2.3, we exhibit r-regular graphs of irregularity strength $\lceil n/r \rceil + 1$. First we determine the strength of cycles C_t ($t \geqslant 3$).

$$\begin{array}{ccccccccccc}
0 & 1 & 2 & 1 & 3 & 1 & 3 & 1 & 3 & \cdots \\
1 & 0 & 3 & 1 & 3 & 1 & 3 & 1 & 3 & \cdots \\
2 & 3 & 0 & 1 & 3 & 1 & 3 & 1 & 3 & \\
1 & 1 & 1 & 0 & 3 & 1 & 3 & 1 & 3 & \\
3 & 3 & 3 & 3 & 0 & 1 & 3 & 1 & 3 & \\
1 & 1 & 1 & 1 & 1 & 0 & 3 & 1 & 3 & \\
3 & 3 & 3 & 3 & 3 & 3 & 0 & 1 & 3 & \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 3 & \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 0 & \\
\vdots & \vdots & & & & & & & & \ddots
\end{array}$$

Fig. 1. Infinite array showing that $s(K_t) \leqslant 3$, $t \geqslant 3$.

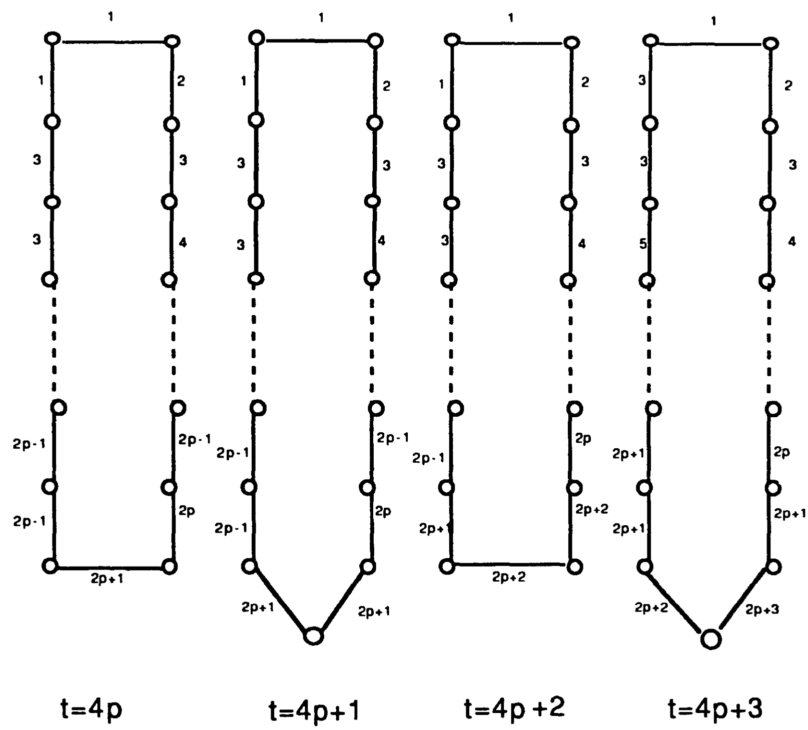

Fig. 2. Irregular networks N_t on C_t having minimum strength.

Theorem 2.6.

$$s(C_t)=\begin{cases}\lceil t/2\rceil & \textit{for } t=4p+1\\ \lceil t/2\rceil+1 & \textit{otherwise.}\end{cases} \tag{2}$$

Proof. From (1)

$$s(C_t)\geqslant\begin{cases}\lceil t/2\rceil & \text{for } t \text{ odd}\\ \lceil t/2\rceil+1 & \text{otherwise}\end{cases}$$

follows. In case $r=4p+3$ this lower bound can be increased by 1. Indeed, suppose to the contrary that there is an irregular network of strength $s=\lceil t/2\rceil=2p+2$ so that the set of degrees in the network are $\{2,3,\ldots,4p+4\}$. Then the sum of all degrees is odd, since there are $2p+1$ odd numbers in the degree set. That contradicts the fact that the total degree sum is twice the sum of the weights in the network.

Irregular networks of minimum strength on C_t are given in Fig. 2. □

Proposition 2.7. *The complete bipartite graph $K_{2k,2k}$ has irregularity strength* 3.

Proof. Obviously $K_{2k,2k}$ is isomorphic to $C_4^{(k)}$. Then $s(K_{2k,2k})=3$ follows from Theorems 2.6, 2.3 and Proposition 2.5. □

This observation (first made in [1]) is extended in Section 4 of the present paper to arbitrary complete bipartite graphs. We conclude this section with the following result.

Theorem 2.8. *Let G be an r-regular graph with n vertices obtained from C_t by vertex multiplication, that is, $G = C_t^{(k)}$ with $t \geq 3$, $n = tk$, $r = 2k$ and $k \geq 2$. Then*

$$s(G) = \lceil n/r \rceil + 1.$$

Proof. From (1) it follows that

$$s(G) \geq \lceil (n + r - 1)/r \rceil = \lceil t/2 \rceil + 1.$$

By applying Theorems 2.6 and 2.3, we show that

$$s(G) \leq \lceil t/2 \rceil + 1. \tag{3}$$

It $t = 4p + 1$, then $s(C_t) = \lceil t/2 \rceil$; if $t = 4p$ or $4p + 3$, then $s(C_t) = \lceil t/2 \rceil + 1$ and N_t (Fig. 2) does not use the maximum possible weight sum. In all of these cases (3) follows as a consequence of Theorem 2.3.

In case $t = 4p + 2$ we give a direct construction of an irregular network of strength $\lceil n/r \rceil + 1$ on $C_t^{(k)}$. Define a $2p + 1 \times 2p + 1$ incident matrix M to represent the weighted edges of the network N_{4p+2} given in Fig. 2 as follows.

Let $M(1, 1) = 1$, $M(2p, 2p + 1) = M(2p + 1, 2p + 1) = 2p + 2$ $M(i, i - 1) = 1$ for $2 \leq i \leq 2p + 1$,

$$M(i, i + 1) = \begin{cases} i & \text{if } i \text{ is odd} \\ i + 1 & \text{if } i \text{ is even} \end{cases} \quad \text{for } 1 \leq i \leq 2p - 1$$

and $M(i, j) = 0$ in all other cases.

Now replace each zero entry of M by a $k \times k$ zero matrix and each $M(u, v) \neq 0$ by a $k \times k$ matrix $M_{u,v}^{(k)}$ defined as follows. If $M(u, v)$ is even, then

$$M_{u,v}^{(k)}(i, j) = \begin{cases} M(u, v) - 1 & \text{for } i + j \leq k \\ M(u, v) & \text{otherwise.} \end{cases}$$

If $M(u, v)$ is odd, then

$$M_{u,v}^{(k)}(i, j) = \begin{cases} M(u, v) & \text{for } i + j \leq k + 1 \\ M(u, v) + 1 & \text{otherwise.} \end{cases}$$

The $k(2p + 1) \times k(2p + 1)$ matrix just described will be modified according to the parity of k as follows. When k is odd, reduce all but the first entry of row $(k - 1)/2$ by 1 in each submatrix $M_{2p+1,2p}^{(k)}$. When k is even reduce by 1 all entries in column $k/2$ of the submatrix $M_{2p,2p+1}^{(k)}$.

In both cases we obtain a totally irregular matrix. The verification (which is routine) is left to the reader. (Figs 3 and 4 show examples for $t = 6$.) $\square$

1	1	1	1	1	1	1	1	1	1						10
1	1	1	1	2	1	1	1	1	2						12
1	1	1	2	2	1	1	1	2	2			O			14
1	1	2	2	2	1	1	2	2	2						16
1	2	2	2	2	1	2	2	2	2						18
1	1	1	1	2						3	3	3	3	4	22
1	1	1	2	2						3	3	3	4	4	24
1	1	2	2	2			O			3	3	4	4	4	26
1	2	2	2	2						3	4	4	4	4	28
2	2	2	2	2						4	4	4	4	4	30
					3	3	3	3	3	3	3	3	3	4	31
					3	2	2	2	3	3	3	3	4	4	29
		O			3	3	3	4	4	3	3	4	4	4	35
					3	3	4	4	4	3	4	4	4	4	37
					3	4	4	4	4	4	4	4	4	4	39
11	13	15	17	19	20	21	23	25	27	32	34	36	38	40	

Fig. 3. Totally irregular 15×15 matrix with maximum entry 4 in the proof of Theorem 2.8 ($p = 1$, $t = 6$ and $k = 5$).

3. The irregularity strength of 2-regular graphs

It is proved in [4] that $(n + r - 1)/r \leqslant s(G) \leqslant n$ for every r-regular graph G with n vertices. We show in this section that $s(G) \leqslant [n/2] + 2$ for 2-regular graphs and use this result to improve the general upper bound in the case of even r.

A 2-regular graph is the disjoint union of cycles. In constructing irregular networks on 2-regular graphs we introduce operations (α) and (β) that split the cycle networks N_t as described in Fig. 2 into smaller cycles. These operations are defined so that the set of degrees of N_t and its strength remain unchanged. Suppose that vertices A, B, C and D generate edges AB and CD of some cycle. If $\omega(AB) = \omega(CD) = a$, then

(α) replace these edges by AC and BD both having weight a (see Fig. 5);

If $\omega(AB) + 1 = \omega(CD)$, then

(β) replace these edges by AC and BD so that $\omega(AC) = \omega(CD)$ and $\omega(BD) = \omega(AB)$. Furthermore, adjust weights belonging to the adjacent four edges of the cycle, as shown in Fig. 6, so that the set of degrees in the network remain the same.

1	1	1	1	1	1	1	1	1	1	1	1							12
1	1	1	1	1	2	1	1	1	1	1	2							14
1	1	1	1	2	2	1	1	1	1	2	2				O			16
1	1	1	2	2	2	1	1	1	2	2	2							18
1	1	2	2	2	2	1	1	2	2	2	2							20
1	2	2	2	2	2	1	2	2	2	2	2							22
1	1	1	1	1	2							3	3	2	3	3	4	25
1	1	1	1	2	2							3	3	2	3	4	4	27
1	1	1	2	2	2				O			3	3	2	4	4	4	29
1	1	2	2	2	2							3	3	3	4	4	4	31
1	2	2	2	2	2							3	4	3	4	4	4	33
2	2	2	2	2	2							4	4	3	4	4	4	35
						3	3	3	3	3	3	3	3	3	3	3	4	37
						3	3	3	3	3	4	3	3	3	3	4	4	39
			O			3	3	3	3	4	4	3	3	3	4	4	4	41
						3	3	3	4	4	4	3	3	4	4	4	4	43
						3	3	4	4	4	4	3	4	4	4	4	4	45
						3	4	4	4	4	4	4	4	4	4	4	4	47
13	15	17	19	21	23	24	26	28	30	32	34	38	40	36	44	46	48	

Fig. 4. Totally irregular 18×18 matrix with maximum entry 4 in the proof of Theorem 2.8 ($p = 1$, $t = 6$ and $k = 6$).

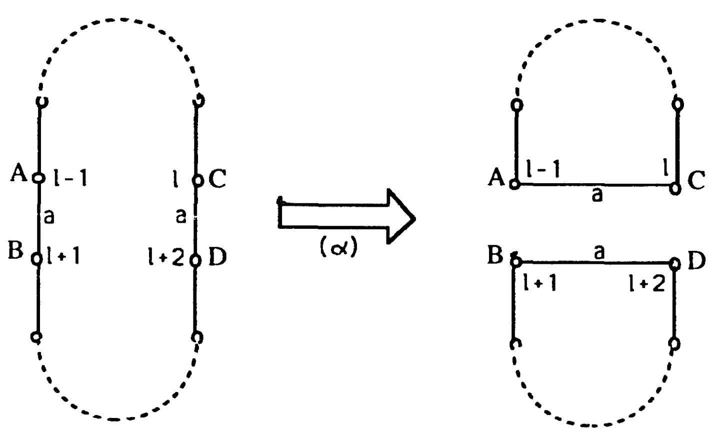

Fig. 5. Operation (α).

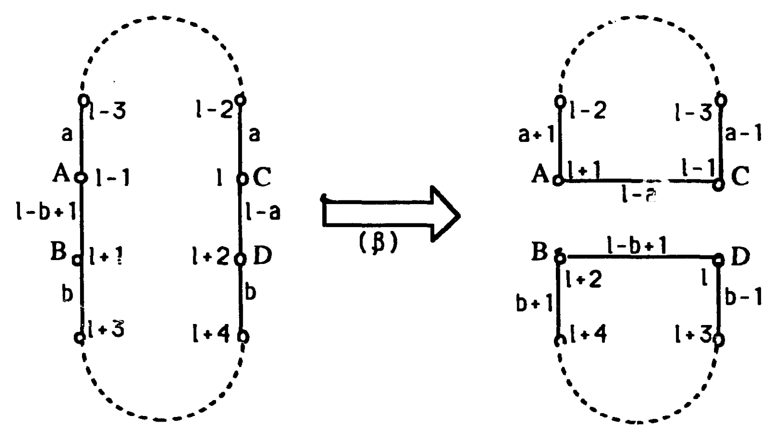

Fig. 6. Operation (β).

Theorem 3.1. *Let G be a 2-regular graph with $4p$ vertices. If G has no triangle components, then its strength is $2p+1$. Moreover, there exists an irregular network of strength $2p+1$ on G with degree set $\{2, 3, \ldots, 4p+1\}$.*

Proof. The desired network will be constructed by induction on the number of components of G.

Step A. If $G = C_{4p}$, then N_t with $t = 4p$ in Fig. 2 is an appropriate irregular network.

Step B. Suppose that G is the disjoint union of the 2-regular graphs G_1 and G_2 having $4p_1$ and $4p_2$ vertices, respectively. Assume that there are iregular networks N_i of strength $2p_i + 1$ on G_i with degree set $\{2, 3, \ldots, 4p_i + 1\}$, for $i = 1$ and 2. Consider the disjoint union of N_1 and N_2 and increase each edge weight in N_2 by $2p_1$. The resulting network on G is obviously irregular, its degree set is $\{2, \ldots, 4p_1 + 4p_2 + 1\}$, and the largest edge weight is $2p_1 + 2p_2 + 1$.

Step C. From steps A and B, we only need to consider the cases when G is the union of two, three or four cycles such that no combination of these components gives a proper subgraph of G whose order is a multiple of $4p$.

We use operations (α) and (β) to obtain the desired network from N_t ($t = 4p$) in Fig. 2.

*Case c.*1: $G = C_{t_1} + C_{t_2}$ *and* $t_1 + t_2 = 4p$
In case $t_1 = 4p_1 + 1$ and $t_2 = 4p_2 + 3$, apply (α) to N_{4p}. If $t_1 = 4p_1 + 2$ and $t_2 = 4p_2 + 2$, then apply (β) to split N_{4p}.

*Case c.*2: $G = C_{t_1} + C_{t_2} + C_{t_3}$ *and* $t_1 + t_2 + t_3 = 4p$
First suppose that $t_1 = 4p_1 + 1$, $t_2 = 4p_2 + 1$ and $t_3 = 4p_3 + 2$. Split N_{4p} in Fig. 2 by

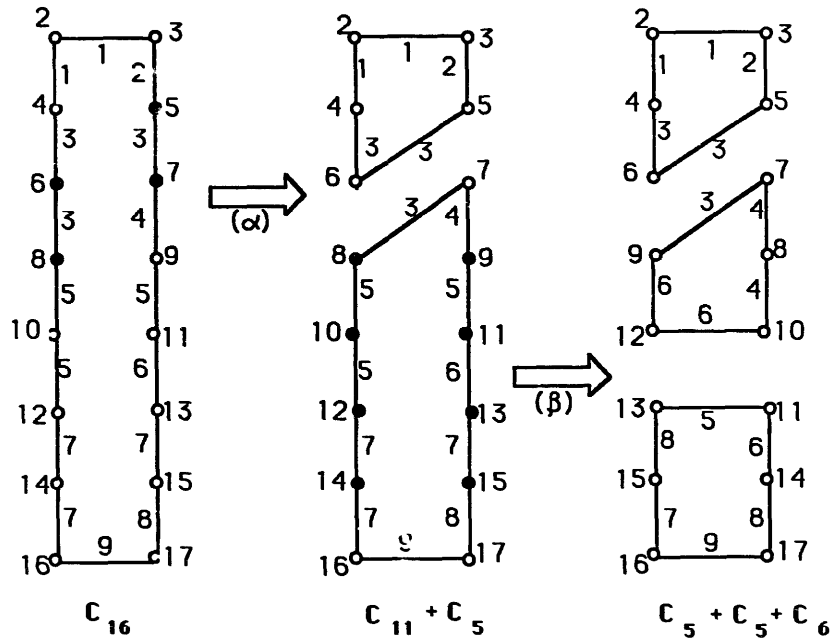

Fig. 7. An irregular network on $C_5+C_5+C_6$.

operation (α) to obtain a network on $C_{t_1}+C_{t_2+t_3}$. Then apply (β) for the sub-network on $C_{t_2+t_3}$ (see the example in Fig. 7.)

In case $t_1=4p_1+2$, $t_2=4p_2+3$ and $t_3=4p_3+3$, first apply (β) on $N_{4(p_1+p_2+1)+1}$. Then add a disjoint copy of N_{4p_3+3} and increase each edge weight by $2(p_1+p_2)+2$. The degree set of the resulting irregular network on $C_{t_1}+C_{t_2}+C_{t_3}$ becomes $\{2,\ldots,4(p_1+p_2)+6\}\cup\{4(p_1+p_2)+7,\ldots,4(p_1+p_2+p_3)+9\}$ and the maximum weight is $2p_3+3+2(p_1+p_2)+2=2p+1$.

Case c.3: $G=\sum_{i=1}^{4} C_{4p_i+1}$ and $\sum_{i=1}^{4}(4p_i+1)=4p$

If at least two cycles have length at least 9, say $p_2, p_3\geqslant 2$, then apply operations (α), (β) and (β) in this order on N_{4p} (see the example in Fig. 8).

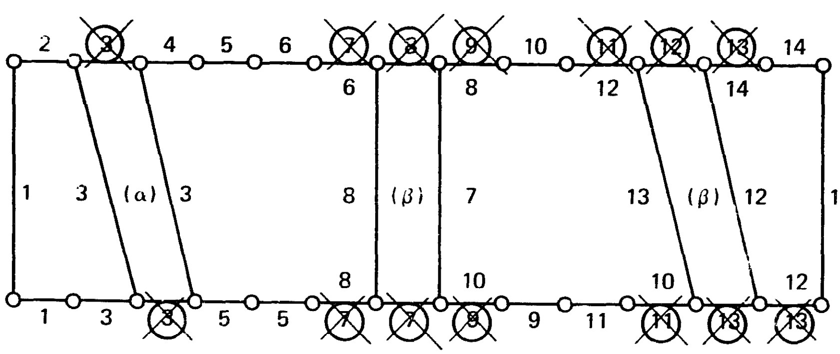

Fig. 8. An irregular network on $G=C_5+C_9+C_9+C_5$ obtained from N_{28} by operations (α), (β) and (β).

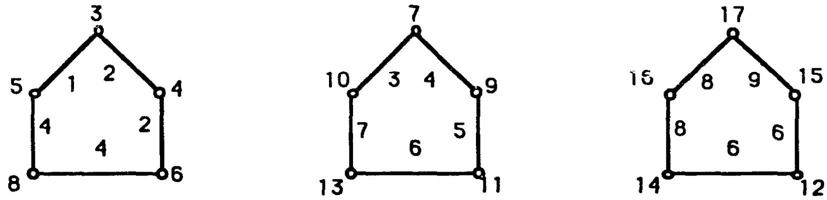

Fig. 9. An irregular network on $C_5 + C_5 + C_5$.

In case $p_2 = p_3 = p_4 = 1$, consider the irregular network on $C_5 + C_5 + C_5$ given in Fig. 9 and increase each edge weight by $2p_1$. Then combine this network with a disjoint copy of N_{4p_1+1}. The network we obtain has strength $2p_1 + 9 = 2p + 1$ and degree set $\{2, \ldots, 4p_1 + 2\} \cup \{4p_1 + 3, \ldots, 4p_1 + 17\}$.

Case c.4: $G = \sum_{i=1}^{4} C_{4p_i+3}$ and $4p = \sum_{i=1}^{4} (4p_i + 3)$
If at least one cycle has length greater than 7, say $p_2 \geq 2$, then take a copy of N_{4q+1} in Fig. 2 (with $q = p_1 + p_2 + p_3 + 2$) and apply (β) twice. In case $P_1 = p_2 = p_3 = 1$, use the network on $C_7 + C_7 + C_7$ defined in Fig. 10.

In both cases the degree set is $\{2, \ldots, 4q + 2\}$. Add a disjoint copy of N_{4p_4+3} and increase each edge weight by $2q$. We obtain an irregular network on G with consecutive degrees and the maximum weight becomes $2q + 2p_4 + 3 = 2p + 1$.

Since cases c.1–c.4 exhaust all possibilities, for Step C, the theorem is proved. □

The proof of Theorem 3.1 can be extended to determine the strength of all 2-regular graphs.

Let N be an irregular network of strength $s = \lceil n/2 \rceil + c$ and consider the set of all possible degrees $D(N) = \{2, \ldots, 2s\}$. Then c is obviously determined by the number, $x(N)$, of unused integers in $D(N)$.

Proposition 3.2. *If N is an irregular network on a 2-regular graph having n vertices, then $s(N) = \lceil n/2 \rceil + \lceil x(N)/2 \rceil$.*

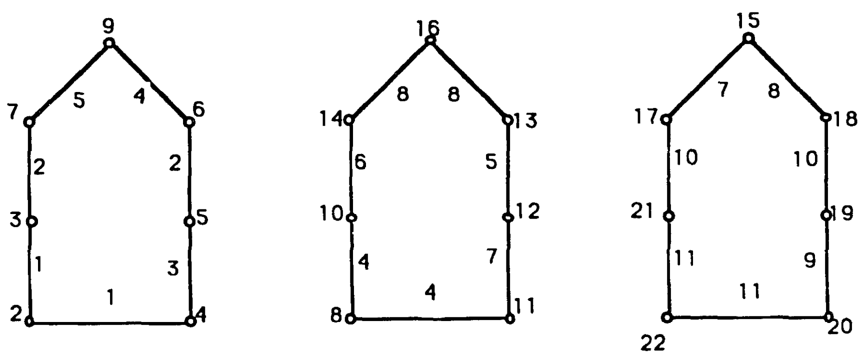

Fig. 10. An irregular network on $C_7 + C_7 + C_7$.

Let us consider first the particular case when each component of the 2-regular graph is a triangle.

Theorem 3.3. *Suppose that G is the disjoint union of triangles and has n vertices. Then*

$$s(G)=\begin{cases}\lceil n/2\rceil+1 & \text{if } n=(4k+3)3\\ \lceil n/2\rceil+2 & \text{otherwise.}\end{cases}$$

Proof. Let N be an irregular network of strength s on G. Weights in the same component are obviously distinct, hence the smallest and largest degree in N is 3 and $2s-1$, respectively. Thus $n\leqslant(2s-1)-3+1$, that is $s\geqslant(n+3)/2$. This is just the lower bound we want to prove except for the case when $n=(4k+3)3$. In this case, however, $D(N)=\{2,\dots,2s\}$ contains $s-1$ odd integers. Since $s-1=(n+3)/2-1=6k+5$ is odd and the total sum of degrees of N is twice the sum of all weights, at least one of the odd integers in $D(N)$ does not belong to the degree set of N. Therefore, $x(N)\geqslant 3$, and by Proposition 3.2, $s\geqslant\lceil n/2\rceil+2$ follows. The existence of the irregular networks having the required strength follows from the following result proved and extended in [3]. □

Lemma. *Let t be a positive integer and*

$$m=\begin{cases}\lceil 3t/2\rceil+1 & \text{if } t=4k+1\\ \lceil 3t/2\rceil+2 & \text{otherwise.}\end{cases}$$

Then there exist t triplets of positive integers not greater than m, $\{a_i, b_i, c_i\}_{i=1}^{t}$ *so that* $\{a_i+b_i, a_i+c_i, b_i+c_i\}_{i=1}^{t}$ *are 3t distinct numbers.*

Fig. 11 shows irregular networks of minimum strength on the disjoint union of t triangles for $t\leqslant 6$ (one has to consider the first t components from left to right).

Theorem 3.4. *Let G be a 2-regular graph of n vertices. Then* $\lceil n/2\rceil\leqslant s(G)\leqslant\lceil n/2\rceil+2$, *moreover if G has no triangle components then*

$$s(G)=\begin{cases}\lceil n/2\rceil & \text{for } n=4k+1\\ \lceil n/2\rceil+1 & \text{otherwise.}\end{cases}$$

Since the proof is of the same flavor as that of Theorem 3.1, we omit details.

Theorem 3.4 can be used to improve the upper bound on the irregularity strength of r-regular graphs, for r even. The best known general upper bound for $s(G)$ is $n-1$ when $n\geqslant 4$ (see [4]). The improved upper bound is given in the next theorem.

Theorem 3.5. *Let G be an r-regular graph with n vertices. If r is even, then* $s(G)\leqslant\lceil n/2\rceil+2$.

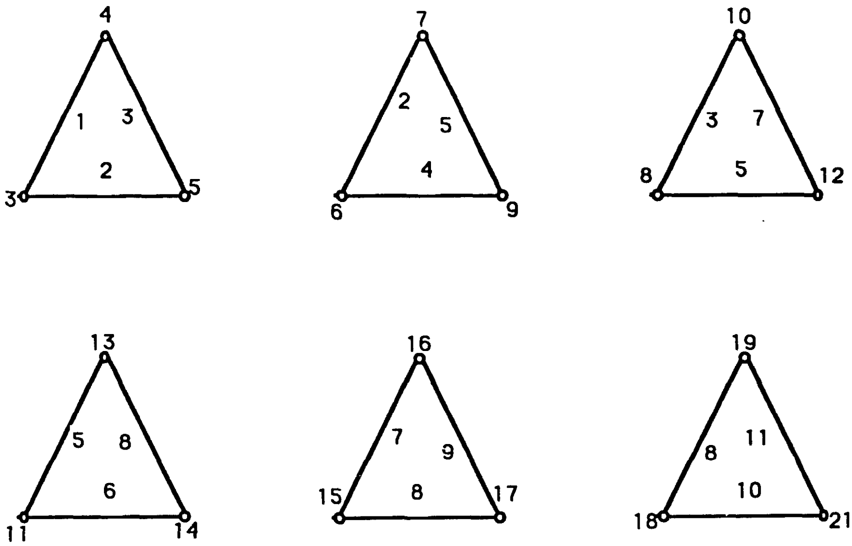

Fig. 11. Irregular networks of minimum strength on $t \cdot C_3$, for $1 \leqslant t \leqslant 6$.

Proof. From Petersen's well-known theorem ([5]), if r is even, then G has a 2-regular spanning subgraph G'. Take an irregular network of minimum strength on G', and assign weight 1 to each edge of $G - G'$. Clearly, we obtain an irregular network on G having strength equal to $s(G')$. By Theorem 3.4, $s(G) \leqslant s(G') \leqslant \lceil n/2 \rceil + 2$ follows. □

It is worth noting that no analogous result is known in case r is odd. We conjecture that the irregularity strength of r-regular graphs of order n is near n/r. However, we do not know the answer for the following weaker question, even in case $r = 3$. For r odd, does there exist a constant c_r, $1/r < c_r < 1$, such that $s(G) \leqslant c_r n$ for every r-regular graph on n vertices?

We conclude this section with a result which does not depend on the parity of r.

Theorem 3.6. *Let G be an r-regular graph with n vertices. If $r \geqslant n/2$, then* $s(G) \leqslant \lceil n/2 \rceil + 1$.

Proof. Since G contains a Hamiltonian cycle C_n according to a classical theorem of Dirac ([2]), we can apply Theorem 2.6 to obtain an irregular network of strength at most $\lceil n/2 \rceil + 1$ on C_n. Then, assign weight one to each edge of $G - C_n$. □

4. The strength of complete bipartite graphs

Let $K_{p,q}$ denote the complete bipartite graph with p and q vertices in its vertex classes, $p \leqslant q$. An irregular network of strength s on $K_{p,q}$ corresponds to a *totally* *irregular* $p \times q$ matrix with distinct row and column sums containing positive integral entries not exceeding s. Since there are q distinct column sums lying between p and sp, $q \leqslant sp - p + 1$. This implies easily that $s \geqslant 3$ and gives the lower bound of the next proposition.

Proposition 4.1. $s(K_{p,q}) \geqslant \max\{3, \lceil (q+p-1)/p \rceil\}$.

This lower bound is the strength of $K_{p,q}$ in most cases.

Lemma 4.2. *Let $q = tp + z$ with $0 \leqslant z < p$ and $t \geqslant 2$. Then there exist irregular $p \times q$ matrices with maximum entry s such that*

$$s \leqslant \begin{cases} t+1 & \text{for } z = 0 \text{ and } 1 \\ t+2 & \text{otherwise.} \end{cases}$$

Proof. Consider the composition of $t+1$ matrices

$$A = [A_1 \mid A_2 \mid \cdots \mid A_t \mid A'_{t+1}],$$

where each A_k, $1 \leqslant k \leqslant t+1$, is a $p \times p$ matrix defined by

$$A_k(i,j) = \begin{cases} k & \text{for } i + j \leqslant p+1 \\ k+1 & \text{otherwise.} \end{cases}$$

and A'_{t+1} is the $p \times z$ submatrix of A_{t+1} consisting of its first z columns. Column sums of A are the integer values from p to $p(t+1) + z - 1$. Row sums are also distinct and the smallest one (that of the first row) is

$$p(1+2+\cdots+t) + z(t+1) = pt(t+1)/2 + z(t+1)$$

Thus for $t \geqslant 2$, the largest column sum is clearly less than any row sum. Hence A is irregular. Furthermore, the largest entry of A is

$$s = A(p,q) = \begin{cases} A_t(p,p) & \text{if } z = 0 \\ A_{t+1}(p,1) & \text{if } z = 1 \\ A_{t+1}(p,z) & \text{otherwise,} \end{cases}$$

which proves the lemma. □

Proposition 4.1 and Lemma 4.2 have the following immediate corollary.

Theorem 4.3. *The irregularity strength of the complete bipartite graph $K_{p,q}$ with $q \geqslant 2p$ is $\lceil (q+p-1)/p \rceil$.*

Now we discuss the remaining cases.

Lemma 4.4. *Let $q/2 \leqslant p < q$. Then there exist irregular $p \times q$ matrices containing only entries 1, 2 and 3.*

Proof. Depending on the parity of q, slightly different constructions are given.

Case (a): $q = 2k$

Define for every $t = 0, 1, \ldots, k$ a $(k+t)\times 2k$ matrix $M_t = A_t + B_t$ as follows. Let

$$A_t(i,j) = \begin{cases} 3 & \text{if } t+1 \leq i \leq k+t \text{ and } 2k+t+1-i \leq j \leq 2k \\ 2 & \text{if } t+1 \leq i \leq k+t \text{ and } k+t+1 \leq i+j \leq 2k+t \\ 1 & \text{otherwise;} \end{cases}$$

and

$$B_t(i,j) = \begin{cases} 1 & \text{if } 1 \leq i \leq k+t \text{ and } 2k+1 \leq i+j \leq 2k+t \text{ or} \\ & \quad t+1 \leq i \leq 2t-1 \text{ and } k-t+1 \leq j \leq k+t-i \\ 0 & \text{otherwise.} \end{cases}$$

Since $A_t(i,j) = 3$ implies that $i+j \geq 2k+t+1$ in which case $B_t(i,j) = 0$, $M_t(i,j) = A_t(i,j) + B_t(i,j) \leq 3$ follows for every i, j.

Figs 12 and 13 show matrices A_t and B_t, respectively. In Fig. 14, M_t is shown for $k = 7$ and $t = 4$.

Now we show that M_t is always irregular. The first $k-t$ column sums, then the first t row sums and the remaining column sums, in that order, are consecutive integers from $t+k+1$ to $2t+3k$. Furthermore, as one can check easily, the last k row sums are distinct integers greater than $2t+3k$.

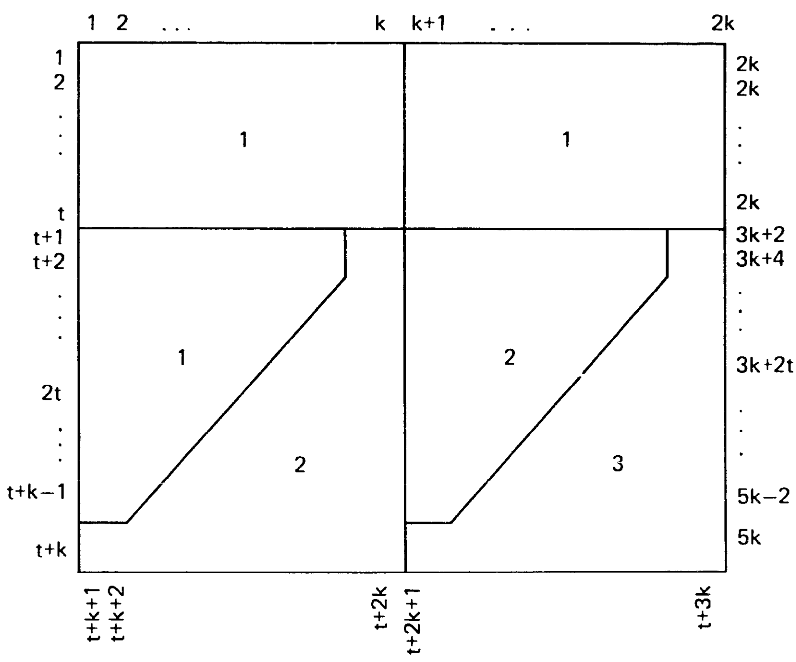

Fig. 12. Matrix A_t in Lemma 4.4.

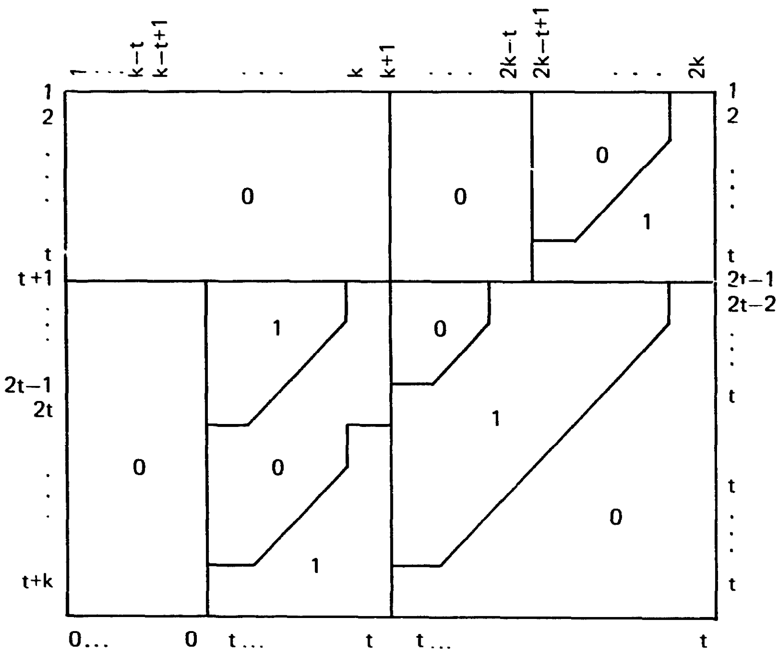

Fig. 13. Matrix B_t in Lemma 4.4.

1	1	1	1	1	1	1	1	1	1	1	1	1	2	15
1	1	1	1	1	1	1	1	1	1	1	1	2	2	16
1	1	1	1	1	1	1	1	1	1	1	2	2	2	17
1	1	1	1	1	1	1	1	1	1	2	2	2	2	18
1	1	1	2	2	2	2	2	2	3	3	3	3	3	30
1	1	1	2	2	2	2	2	3	3	3	3	3	3	31
1	1	1	2	2	2	2	3	3	3	3	3	3	3	32
1	1	1	2	2	2	3	3	3	3	3	3	3	3	33
1	1	2	2	2	3	3	3	3	3	3	3	3	3	35
1	2	2	2	3	3	3	3	3	3	3	3	3	3	37
2	2	2	3	3	3	3	3	3	3	3	3	3	3	39
12	13	14	19	20	21	22	23	24	25	26	27	28	29	

Fig. 14. A totally irregular 11×14 matrix.

$$
\begin{array}{c|ccc|c|ccc|ccccccc|c}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 16 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 17 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 18 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 19 \\
\hline
1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 30 \\
1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 & 31 \\
1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 32 \\
1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 34 \\
1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 36 \\
1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 38 \\
1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 39 \\
\hline
11 & 12 & 13 & 14 & 15 & 20 & 21 & 22 & 23 & 24 & 25 & 26 & 27 & 28 & 29 &
\end{array}
$$

Fig. 15. A totally irregular 11×15 matrix.

Case (b): $q = 2k + 1$

For every $t = 1, 2, \ldots, k$, we modify matrices A_t and B_t defined above as follows. Add as a new first column to A_t a column of ones. Add a new first column of zeros to B_t and set each entry of the $(k - t + 1)$st column of B_t to zero. The sum of these matrices $M'_t = A'_t + B'_t$ is a totally irregular $(t + k) \times (2k + 1)$ matrix with maximal entry 3. Details are left to the reader. In Fig. 15, M'_t is given for $k = 7$ and $t = 4$. □

Proposition 4.1 and Lemma 4.4 have the following immediate corollary.

Theorem 4.5. *The irregularity strength of the complete bipartite graph* $K_{p,q}$ *with* $1 < q/2 \leqslant p < q$, *is equal to* 3.

The study of the irrregularity strength of the complete bipartite graph $K_{n,n}$ was proposed by Chartrand et al. in [1]. It is known that $s(K_{n,n}) = 3$ for n even (see Proposition 2.7). However, the question whether $s(K_{2k+1,2k+1}) = 4$ is still open. We restate here this challenging conjecture in terms of matrices.

Conjecture 4.6. If a $(2k + 1) \times (2k + 1)$ matrix has entries -1, 0 and 1, then there are identical values among the $4k + 2$ row and column sums.

Addendum. It has recently been shown by A. Gŷarfas that $s(K_{2k+1,2k+1}) = 4$ for all $k \geqslant 1$.

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